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REGEXES, KLEENE ALGEBRAS, AND REAL ULTIMATE POWER!

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WHO AM I?

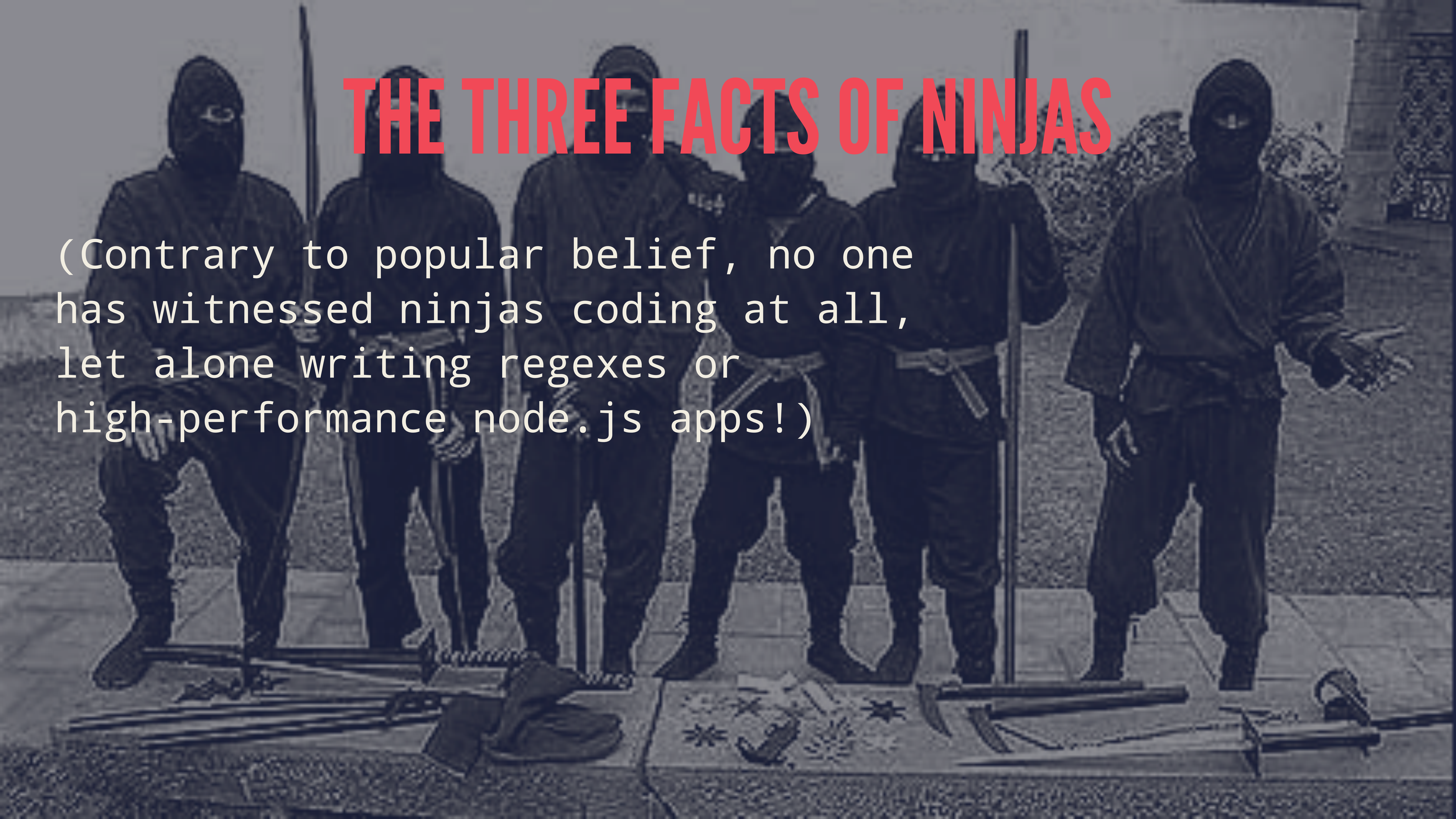
- typelevel member λ
- maintain spire and several other scala libraries
- interested in expressiveness and performance ☯
- hack scala code at meetup

code at http://github.com/non

THE THREE FACTS OF REGEXES

1. Regexes are finite automatons.
2. Regexes backtrack ALL the time.
3. The purpose of a regex is to flip out and kill people.

http://www.realultimatepower.net/index4.htm



THE THREE FACTS OF NINJAS

(Contrary to popular belief, no one has witnessed ninjas coding at all, let alone writing regexes or high-performance node.js apps!)

OVERIEW

regular expressions (aka regexps or regexes)

- formalized by Stephen Kleene (1956)
- popularized in unix (grep, ed, sed, awk, etc.)
- mainstay of modern programming
- provided as syntax (e.g. perl, ruby, js, php) or library

EXAMPLES

positive integer

```
[1-9][0-9]*
```

string with escaping

```
"([^"\\]|\\.)*"
```

source file name

```
.*\.(clj|erl|hs|idr|scala|scm|sml)
```

Some people, when confronted
with a problem, think "I know,
I'll use regular expressions."
Now they have two problems.

Jamie Zawinski, 1997

alt.religion.emacs

REGEX PROBLEMS

1. use with non-regular grammers (parsing with regexes)
2. coercing arbitrary data into byte-streams (the unix problem)
3. terse, "unreadable" regex syntax
4. extended features (the perl problem)

PERL.JPG

```perl
my $var = 'testing';
$_ = 'In this string we are $var the "e" modifier.';

s/(\$\w+)/$1/ee;

print; # prints "In this string we are testing the "e" modifier."
```

It's easy to try to fix "regex problems" with more regexes.

This is almost always the wrong thing to do. ☹

So why is this talk about regular expressions anyway?

GOOD THINGS ABOUT REGEXES

1. composable
2. deterministic and easy to test
3. total (although can be very slow)
4. pure (side effects are impossible)

(...but any of these can be broken by "extended features".)

We will explore the power of regular expressions by generalizing them to a more abstract concept:

Kleene algebras

Credit for this idea belongs to Russell O'Connor, who wrote "A Very General Method of Computing Shortest Paths" in literate Haskell.

IMPLEMENTING REGULAR EXPRESSIONS

Let's create a simple **AST** (abstract syntax tree) which can represent basic regular expressions.

We'll use **sealed traits**, and separate **data** from **logic**.

AST IMPLEMENTATION (1 OF 2)

```scala
sealed trait Expr[+A]

// matches nothing, represents ∅ (the null set).
case object Nul extends Expr[Nothing]

// matches the empty string (ε).
case object Empty extends Expr[Nothing]

// matches a literal value (a).
case class Var[A](a: A) extends Expr[A]
```

AST IMPLEMENTATION (2 OF 2)

```
// matches x or y, represents (x|y).
case class Or[A](x: Expr[A], y: Expr[A]) extends Expr[A]

// matches x and y, represents (xy).
case class Then[A](x: Expr[A], y: Expr[A]) extends Expr[A]

// matches zero-or-more xs, represents (x*).
case class Star[A](x: Expr[A]) extends Expr[A]
```

WEIRD REQUEST

In the next slides, we'll introduce some algebra.

In the context of regexes, let's imagine that:

$(x + y)$ means the regex (x|y)
$(x \cdot y)$ means the regex (xy)

(It may help to think of sum vs. product types.)

ALGEBRA DETOUR

Let's talk about algebras.

An **algebra** is just a set (e.g. A) together with operations on that set. For example:

$$(A + A) \to A$$

$$(-A) \to A$$

Let's define a **Rig** as an algebra with the following:

$$(A + A) \to A$$

$$(A \cdot A) \to A$$

$$0 \text{ such that } (x + 0) = x$$

$$1 \text{ such that } (x \cdot 1) = x \text{ and } (x \cdot 0) = 0$$

Both operators are associative, and + is commutative.

Type class implementation:

```
trait Rig[A] {
  def zero: A
  def one: A
  def plus(x: A, y: A): A
  def times(x: A, y: A): A
}
```

(See spire.algebra.Rig)

By adding a kstar operate to our Rig we get a Kleene Algebra.
We can also define a kplus operator in terms of kstar.

Again, let's imagine that:

x.kstar means the regex x*
x.kplus means the regex x+
(x.kplus = x · x.kstar)

There are four laws that our **Kleene Algebra** must obey.
They make can be understood in terms of our regex assumptions.

$$x^* = \varepsilon | xx^* = \varepsilon | x^*x$$

$$x | x = x$$

$$ax | x = x \Rightarrow a^*x | x = x$$

$$xa | x = x \Rightarrow xa^* | x = x$$

(If only the first law holds we call it a **Star Rig** instead.)

IMPLEMENTATION TIME!

```
trait ExprHasKleene[A] extends Kleene[Expr[A]] {
  def zero = Nul
  def one = Empty

  // implementation continued...
}
```

OPERATORS (1 OF 3)

```
// x + y is equivlant to x|y
def plus(x: Expr[A], y: Expr[A]): Expr[A] =
  (x, y) match {
    case (Nul, e) => e
    case (e, Nul) => e
    case (Empty, Empty) => Empty
    case (Empty, Star(e)) => Star(e)
    case (Star(e), Empty) => Star(e)
    case (e1, e2) => Or(e1, e2)
  }
```

OPERATORS (2 OF 3)

```
// x * y is equivlant to xy
def times(x: Expr[A], y: Expr[A]): Expr[A] =
  (x, y) match {
    case (Nul, _) => Nul
    case (_, Nul) => Nul
    case (Empty, e) => e
    case (e, Empty) => e
    case (e1, e2) => Then(e1, e2)
  }
```

OPERATORS (3 OF 3)

```
// x.kstar is equivlant to x*
def kstar(x: Expr[A]): Expr[A] =
  x match {
    case Nul => Empty
    case Empty => Empty
    case Star(e) => kstar(e)
    case x => Star(x)
  }
```

MATCHING

Let's look at some ways to use our AST to match input.

This is just a first pass, to be sure that our types are expressive enough. We can optimize later.

Notice that there is no mention of Char or String.

```scala
def matches[A](expr: Expr[A], input: IndexedSeq[A]): Boolean = {

  def fits(pos: Int, a: A) = pos < input.length && input(pos) == a

  def look(expr: Expr[A], pos: Int): Stream[Int] =
    expr match {
      case Nul => Stream.empty
      case Empty => Stream(pos)
      case Var(a) if fits(pos, a) => Stream(pos + 1)
      case Var(_) => Stream.empty
      case Or(x, y) => look(x, pos) ++ look(y, pos)
      case Then(x, y) => look(x, pos).flatMap(look(y, _))
      case Star(x) => pos #:: look(x, pos).flatMap(look(expr, _))
    }
  look(expr, 0).exists(_ == input.length)
}
```

LAZY MATCHING

Instead of matching a fixed size collection, let's try matching a stream.

(We'll use Stream[A] but any lazy, sequential data structure would work.)

```scala
def smatches[A](expr: Expr[A], input: Stream[A]): Boolean = {

  def fits(s: Stream[A], a: A) = s.headOption.map(_ == a).getOrElse(false)

  def look(expr: Expr[A], input: Stream[A]): Stream[Stream[A]] =
    expr match {
      case Nul => Stream.empty
      case Empty => Stream(input)
      case Var(a) if fits(input, a) => Stream(input.tail)
      case Var(_) => Stream.empty
      case Or(x, y) => look(x, input) ++ look(y, input)
      case Then(x, y) => look(x, input).flatMap(look(y, _))
      case Star(x) => input #:: look(x, input).flatMap(look(expr, _))
    }

  Look(expr, input).exists(_.isEmpty)
}
```

KICKING THE TIRES

```
// this assumes that our AST, our type class instance,
// implicit operators, and matches/smatches are in scope
def v[A](a: A) = Var(a)

// wo+(w|t)
val p = v('w') * v('o').kplus * (v('w') + v('t'))
p.matches("wow")    // true
p.matches("woooow") // true
p.matches("woot")   // true
p.matches("wood")   // false
```

KICKING THE TIRES HARDER

```
// we can also match more interesting things
val bytes: Array[Byte] = ...
val header = bytes.take(8)

// we could add a Dot object to Expr[A]
val dot = v(0.toByte) + v(1.toByte) + ...
val fourBytes = dot * dot * dot * dot
val zero = v(0.toByte)
val fortyTwo = v(42.toByte)

val intel = v('I'.toByte) * v('I'.toByte) * fortyTwo * zero * fourBytes
val motoral = v('M'.toByte * v('M'.toByte) * zero * fortyTwo * fourBytes
val tiff = intel + motorola
```

OBSERVATIONS

1. AST construction is ugly, but easy to improve.
2. We can name and compose regex fragments.
3. Matching generalizes to bytes, sequences or numbers, etc.
4. We can extend our AST to support richer constructs.

WHAT NOW?

So we created a novel regex implementation.

Is that it?

NO!

THE MATRIX

Let's start with this abstract definition of a square matrix.

```
case class Dim(n: Int)

trait Matrix[A] { lhs =>
  def dim: Dim
  def apply(x: Int, y: Int): A
  def map[B: ClassTag](f: A => B): Matrix[B]
  def +(rhs: Matrix[A])(implicit rig: Rig[A]): Matrix[A]
  def *(rhs: Matrix[A])(implicit rig: Rig[A]): Matrix[A]
}
```

THE MATRIX, RELOADED

We'll use apply to build matrix instances.

```
object Matrix {
  def apply[A: ClassTag](f: (Int, Int) => A)(implicit dim: Dim): Matrix[A] = ...

  def zero[A: Rig: ClassTag](implicit dim: Dim): Matrix[A] =
    apply((x, y) => Rig[A].zero)

  def one[A: Rig: ClassTag](implicit dim: Dim): Matrix[A] =
    apply((x, y) => if (x == y) Rig[A].one else Rig[A].zero)
}
```

THE DOORS OF PERCEPTION

Let's think about a matrix as a **graph** or **state machine**. Each row represents **transitions**, and each operation produces a new state machine.

A + B takes a transition from **A** or **B** (**a|b**).
A · B takes a transition from **A** then from **B** (**ab**).

Using this insight we can define an algebra for matrices.

TAKE THE BLUE PILL

```
implicit def matrixHasKleene[A: Kleene: ClassTag](implicit dim: Dim) =
  new Kleene[Matrix[A]] {
    def zero: Matrix[A] = Matrix.zero
    def one: Matrix[A] = Matrix.one
    def plus(x: Matrix[A], y: Matrix[A]) = x + y
    def times(x: Matrix[A], y: Matrix[A]) = x * y

    def kstar(m: Matrix[A]) = zero + kplus(m)

    def kplus(m: Matrix[A]) = ...
  }
```

TAKE THE RED PILL

```
// this is essentially a generic implementation of
// the floyd-warshall algorithm
def kplus(m: Matrix[A]) = {
  def f(k: Int, m: Matrix[A]) = Matrix[A] { (x, y) =>
    m(x, y) + m(k, y) * m(k, k).kstar * m(x, k)
  }
  @tailrec def loop(m: Matrix[A], i: Int): Matrix[A] =
    if (i >= 0) loop(f(i, m), i - 1) else m
  loop(m, dim.n - 1)
}
```

DON'T PANIC!

A.kplus expands to A + A * A.kplus

(A + AA + AAA + ...)

Rather than summing an infinite series, we can just compute the transitive closure, which will be equivalent under the assumptions of our Kleene[A] algebra.

This next bit is just definining tons of cool types to parameterize our matrix with. We'll go a bit fast.

⚠ Buckle up! ⚠

GRAPH TYPES

```
case class Edge(from: Int, to: Int)

object Graph {
  def apply(edges: Edge*)(implicit dim: Dim): Matrix[Boolean] = ...
}

object LabeledGraph {
  def apply(m: Matrix[Boolean])(implicit dim: Dim) =
    Matrix[Option[Edge]] { (x, y) =>
      if (m(x, y)) Some(Edge(y, x)) else None
    }
}
```

EXAMPLE GRAPH

```
val edges = List(
  Edge(0, 1), Edge(1, 2), Edge(1, 3),
  Edge(2, 4), Edge(3, 1), Edge(4, 2))

val example = Graph(edges:_*)(Dim(5))

example          example.kplus   example.kstar
(adjacency)      (t. closure)    (reflective t. closure)
. x . . .        . x x x x       x x x x x
. . x x .        . x x x x       . x x x x
. . . . x        . . x . x       . . x . x
. x . . .        . x x x x       . x x x x
. . x . .        . . x . x       . . x . x
```

LABELS AND PATHS

```
val labeled: Marix[Edge] = LabeledGraph(example)
val expred: Matrix[Expr[Edge]] =
  labeled.map(_.map(Expr(_)).getOrElse(Nul))

expred:
∅ AB ∅  ∅  ∅
∅ ∅  BC BD ∅
∅ ∅  ∅  ∅  CE
∅ DB ∅  ∅  ∅
∅ ∅  EC ∅  ∅

expred.kstar:
...
```

A→A	ε
A→B	(AB|AB(BDDB)*BDDB)
A→C	AB(BDDB)*(BC|BC(CEEC)*CEEC)
A→D	AB(BDDB)*BD
A→E	AB(BDDB)*BC(CEEC)*CE
B→A	∅
B→B	(ε|(BDDB|BDDB(BDDB)*BDDB))
B→C	((BC|BC(CEEC)*CEEC)|BDDB(BDDB)*(BC|BC(CEEC)*CEEC))
B→D	(BD|BDDB(BDDB)*BD)
B→E	(BC(CEEC)*CE|BDDB(BDDB)*BC(CEEC)*CE)
C→A	∅
C→B	∅
C→C	(ε|(CEEC|CEEC(CEEC)*CEEC))
C→D	∅
C→E	(CE|CEEC(CEEC)*CE)
D→A	∅
D→B	(DB|DB(BDDB)*BDDB)
D→C	DB(BDDB)*(BC|BC(CEEC)*CEEC)
D→D	(ε|DB(BDDB)*BD)
D→E	DB(BDDB)*BC(CEEC)*CE
E→A	∅
E→B	∅
E→C	(EC|EC(CEEC)*CEEC)
E→D	∅
E→E	(ε|EC(CEEC)*CE)

LEAST COST PATH

```
sealed trait Weight[+A]
case class Finite[A](a: A) extends Weight[A]
case object Infinity extends Weight[Nothing]

object Weight {
  def apply[A](a: A): Weight[A] = Finite(a)
  def inf[A]: Weight[A] = Infinity
}
```

LEAST COST PATH

When talking about least cost, we'll define an algebra where

$x + y$ really means $\min(\text{cost}(x), \text{cost}(y))$
$x \cdot y$ really means $\text{cost}(x) + \text{cost}(y)$

This allows an infinite weight to function as 0.

WEIGHTS

```
implicit def WeightHasKleene[A: Order: Rig] = new Kleene[Weight[A]] {
  def zero: Weight[A] = Infinity
  def one: Weight[A] = Weight(Rig[A].zero)
  def plus(x: Weight[A], y: Weight[A]): Weight[A] = (x, y) match {
    case (Infinity, t) => t
    case (t, Infinity) => t
    case (Finite(a1), Finite(a2)) => Weight(a1 min a2)
  }
  def times(x: Weight[A], y: Weight[A]): Weight[A] = (x, y) match {
    case (Infinity, _) | (_, Infinity) => Infinity
    case (Finite(a1), Finite(a2)) => Weight(a1 + a2)
  }
  def kstar(x: Weight[A]): Weight[A] = one
}
```

```scala
implicit val dim = Dim(6)

val edges = List(
  (Edge(0, 1), 7), (Edge(0, 2), 9), (Edge(0, 5), 14),
  (Edge(1, 2), 10), (Edge(1, 3), 15),
  (Edge(2, 3), 11), (Edge(2, 5), 2),
  (Edge(3, 4), 6), (Edge(4, 5), 9))

val weighted: Matrix[Weight[Int]] = ...
```

```
weighted:                weighted.kstar (least cost):
∞  7  9  ∞  ∞ 14         0  7  9  20 20 11
7  ∞  10 15 ∞ ∞          7  0  10 15 21 12
9  10 ∞  11 ∞ 2          9  10 0  11 11 2
∞  15 11 ∞  6 ∞          20 15 11 0  6  13
∞  ∞  ∞  6  ∞ 9          20 21 11 6  0  9
14 ∞  2  ∞  9 ∞          11 12 2  13 9  0
```

IT JUST KEEPS ON GOING...

We can combine paths and least cost to get shortest path, as well as generating an (potentially infinite) stream of all possible paths.

As long as we can define a **Kleene algebra** instance for a type **T**, we can use **Matrix[T].kstar** to "explore" the search space.

OTHER BENEFITS

- Generate stream of all possible matches, or a random match.
- Compile Regex → NFA → DFA
- Hopcroft DFA minimization + regex equality test.
- Determine subset/superset for two regexes (inclusion).
- Compile-time syntax checking

EVEN MORE BENEFITS

Regular expressions can always be compiled to a **DFA** (deterministic finite automata).

In practice many implementations simply use **backtracking**.

This can be... **bad**.

DERP.JPG

How long should a regular expression match take?

```
val s = "0000000000000000000000000000"
val pattern = java.util.regex.Pattern.compile("(0*)*A")
val p: Expr[Char] = v('0').kstar.kstar * v('A')
val m: Dfa[Char] = p.minimize

timer { p.matches(s) }                // 1.063 ms
timer { m.accept(s) }                 // 0.113 ms
timer { pattern.matcher(s).find() }   // 27048.943 ms
```

WHERE CAN I BUY THIS?

An experimental library called irreg is available on Github:

http://github.com/non/irreg

(No packages published yet. More work is needed.)

SHORTCOMINGS / FUTURE WORK

1. wildcard (.) support
2. optimized character ranges (especially for Dfa[A])
3. subgroup matching (and maybe references?)
4. DSL for working with Char and String
5. Dfa[A] combinators (complement, intersection, difference, ...)

BIG IDEAS

1. Finding the right abstract can be amazing.
2. Limiting the scope of an algebra may make it more useful
3. Keeping data separate from logic aids clarity.
4. There is always more to learn.
5. Another world is possible!

THAT'S ALL, FOLKS!

```
def s(xs: String): Expr[Char] =
  xs.map(Var(_)).reduceLeft(_ * _)
def o(e: Expr[Char]): Expr[Char] = e + Empty

val e = o(s("mange ")) * o(s("tusen ")) * s("takk!")

e.matches("takk!")
e.matches("mange takk!")
e.matches("tusen takk!")
e.matches("mange tusen takk!")
```