Intervals:
unifying uncertainty, ranges, and loops

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who am i?

• **typelevel** member \( \lambda \)
• maintain **spire** and several other scala libraries
• interested in expressiveness **and** performance 🌟
• hack scala code at [meetup](http://github.com/non)

code at [http://github.com/non](http://github.com/non)
These slides assume the following imports.

```scala
import spire.algebra._
import spire.implicits._
import spire.math._
```

Together they import all of Spire's types, type classes, and syntax. It's ideal for exploratory work in the REPL.

You can find the code for Spire at:

`https://github.com/non/spire`
what will this talk cover?

- Basic description of the `Interval[_]` type
- Arithmetic, uncertainty, and error propagation
- Set operations and iteration
- Demo (?)
- Conclusion
intervals
what is an interval?

Intervals represent a range of real numbers.

For example: *all numbers between 3 and 6.*

Sound familiar?
what is an interval?

Reminds me of scala.Range:

scala> (3 to 6)
res0: Range.Inclusive = Range(3, 4, 5, 6)

Or maybe:

scala> (3 until 6)
res1: Range = Range(3, 4, 5)
what's the difference?

Range has two modes:

- *inclusive*: (3 to 6) includes 6
- *exclusive*: (3 until 6) excludes 6

(Both cases include 3.)

Our example is *all numbers between 3 and 6*. Should that include 3 and 6?
being precise

The first argument of Range is always *inclusive*, so we wouldn't be able to exclude both end points.

Maybe Range is not the right way to express this idea?
Let's use notation to be a bit clearer about what we mean.

We'll use:

* parenthesis as exclusive: \((3, 6)\)
* square brackets as inclusive: \([3, 6]\)
* we can combine them too: \([3, 6)\)

(Note: this is not valid Scala code.)
special cases

There are some special intervals worth mentioning:

(∅) empty interval (contains nothing)
[1] the single value 1 (degenerate)
(3, ∞) contains numbers > 3
(-∞, 0] contains numbers <= 0
(-∞, ∞) contains all numbers

(The bottom three are called *unbounded* intervals.)
Going back to the range example, we see:

```scala
scala> (3 to 6)
res0: Range.Inclusive = Range(3, 4, 5, 6)

scala> res0.contains(4.5)
res1: Boolean = false
```

(Ignore the fact that this type-checks.)

It seems clear that the interval $[3, 6]$ should contain 4.5.
time for polymorphism!

Range was written in terms of `Int`, to support things like:

```scala
(3 to 6).toList
(3 to 6).map(...)
for (i <- 3 to 6) { ... }
```

(Let's not talk about `NumericRange[_]` right now.)

In our case, we can support `Int`, `Double`, or any other type we want using Spire's type classes.
def length = uncountable

Remember: *intervals represent ranges of real numbers.*

Even $(0, 1)$ has uncountably-many members.

This means we can't effectively model intervals with:

- Range
- Set[_]
- Stream[_]

...or any other Scala collection.
design goals

1. Defined in terms of endpoints -- not a container
2. Support unbounded intervals (above, below, or both)
3. Generic: parameterize the number type

(We want an intensional definition.)
Let's say we're representing \((x, y)\) as an Interval\([A]\).

We know that \(x < y\) (i.e. the lower bound is below the upper).

Therefore, we definitely need to compare \(A\) values.

In Spire, we use \(\text{Order}[A]\) to do this.
interval arithmetic
why bother creating intervals?

- Uncertainty about a value (e.g. $3 \pm 0.1$)
- Work with a set of values efficiently
- Provide approximate answers to harder problems
- Efficiently support distributed calculation
how do we use them?

Mathematically, interval arithmetic is equivalent to:

1. doing the operations over each element
2. building the smallest interval containing the results

For example:

\[
[1, 2] + 1 \\
[1 + 1, \ldots 2 + 1] \\
[2, 3]
\]
a "graphical" view

We can see the same operation in ASCII art:

```
0  1  2  3  4
[1,  2]    [---]
+    1    x
= [2,  3]    [---]
```

By adding 1 to each end point we get the correct result.
more addition

We can add two intervals together just as easily:

\[
\begin{array}{ccccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
[1, 2] & [---] \\
+ [1, 3] & [-------] \\
= [2, 5] & [----------]
\end{array}
\]

We are adding every possible value of \([1, 2]\) to every possible value of \([1, 3]\) to get \([2, 5]\).
It generalizes

This strategy works for *well-behaved* operations, i.e.

- the operation is deterministic
- the range of outputs is determined by end points and/or zero
- we know what to do in unbound cases

Interval[...] supports:

abs vmin vmax + - * / reciprocal pow sqrt nroot
Let's start simple:

```scala
val w = Interval(3.0, 5.0)  // [3.0, 5.0]
val x = Interval.point(2.0)  // [2.0]
val y = 10.0 ± 0.1           // [9.9, 10.1]
val z = Interval.above(0.0)  // (0.0, ∞)
```

```scala
w.isEmpty       // false
x.isPoint       // true
y.contains(10.0) // true
z.contains(0.0) // false
```
**arithmetic**

```scala
val w = Interval(3.0, 5.0)  // [3.0, 5.0]
val x = Interval.point(2.0)  // [2.0]
val y = 10.0 ± 0.1           // [9.9, 10.1]
val z = Interval.above(0.0)  // (0.0, ∞)

w * x                       // [6.0, 10.0]
w * x + y                   // [15.9, 20.1]
(x * y)                     // [19.8, 20.2]
(x * y) vmin z              // (0.0, 20.2]
((y ** 2) * 2).sqrt         // [14.000714267493642, 14.28355697996826]
```
We can create intervals for anything with an order:

```
val a = Interval("bats", "cats")
val b = Interval.below(Option(6))
val c = Interval.empty[List[Long]]

a.contains("batter") // true
b + b // (-\infty, Some(12))
c + List(3L) // (\emptyset)
```
wait...

I thought you said intervals approximated real numbers?

What gives?
those aren't real numbers!

Spire generalizes `Interval[_]` to support all ordered types.

We then restrict specific interval operations.

```scala
val a = Interval("bats", "cats")
val b = Interval.below(Option(6))
val c = Interval.empty[List[Long]]

a * a // does not compile
b.sqrt // does not compile
c / 2L // does not compile
```
type classes are capabilities

The compiler informs us that our types are not sufficiently number-like for the given operations.

(i.e. they don't behave sufficiently like real numbers.)

Here are the missing capabilities:

- `a * a` needs `Semiring[_]`
- `b.sqrt` needs `Ring[_]`, `NRoot[_]`
- `c / List(2L)` needs `Field[_]`
intervals as specialized sets
basic set operations

We've shown how to combine intervals using arithmetic.

But we can also think of intervals as a kind of set:

```scala
def basic_set_operations(): Unit = {
  val p = Interval.above(0) // (0, ∞)
  val q = Interval.below(3) // (-∞, 3)

  p.contains(0) // false
  q.contains(0) // true
}
```
more set operations

val p = Interval.above(0)  // (0, ∞)
val q = Interval.below(3)  // (-∞, 3)

p & q           // (0, 3)
p | q             // (-∞, ∞)
val t = Interval(1, 2)  // [1, 2]
t intersects p  // true
t isSubsetOf q   // true
~p               // List((-∞, 0])
~t               // List((-∞, 1), (2, ∞))
p -- t           // List((0, 1), (2, ∞))
set visualization

\[ p = (0, \infty) \]
\[ q = (-\infty, 3) \]
\[ t = [1, 2] \]

\[ p \& q \]
\[ p \mid q \]
\[ \sim p \]
\[ \sim t \]
going even farther

Some operations (e.g. ~x) return a list of intervals.

Returning a single object would fit the set metaphor better.

If you want this, see Rüdiger Klaehn's intervalset project:

https://github.com/rklaehn/intervalset

(It's possible this structure will be added to a Spire module.)
if it's a set, can we map?

Sort of.

def mapBounds[B: Order](f: A => B) = {
    val lb = lowerBound.map(f)
    val ub = upperBound.map(f)
    Interval.fromBounds(lb, ub)
}

However, this isn't called map because it's not very well-behaved.
functor problems

Easy to pass `mapBounds` functions that violate our expectations.

```scala
val h = Interval(r"-2", r"2") // [-2, 2]

h ** 2 // [0, 4]
val h = Interval(r"-2", r"2") // [-2, 2]

h.mapBounds(x => x ** 2) // [4]

-h // [-2, 2]

h.mapBounds(x => -x) // (Ø)

Is there a better way?
```
For our purposes, polynomials are:

- functions of a single variable (e.g. $x$)
- limited set of operations they can use

$$f(x) = x^2 + 9$$
$$f(x) = 3x^2$$
$$f(x) = 2x^2 + x$$

(Not valid scala code.)
Here's how to encode those polynomials in Spire:

```plaintext
// fancy string interpolation
poly"x^2 + 9" // Polynomial[Rational] = (x^2 + 9)

// sequence of terms
Polynomial(poly.Term(r"3", 2) :: Nil) // (3x^2)

// map of exponents -> coefficients
Polynomial(Map(2 -> r"2", 1 -> r"1")) // (2x^2 + x)
```

(Thanks to James Thompson for this excellent polynomial type.)
what does that do for us?

The clue is in the *limited set of operations polynomials can use.*

(These correspond to the *well-behaved* interval operations.)

```scala
val h = Interval(r"-2", r"2")  // [-2, 2]
val p = poly"x^2 + 9"  // (x^2 + 9)

h.translate(p)  // [9, 13]
```

```scala
case class Int v
```

```scala
def translate(p: pol)
```

```scala
h.translate(poly"-x")  // [-2, 2]
```

```scala
h.translate(poly"x^3 + 3x^2")  // [-8, 20]
```
how does it work?

We turn Polynomial[A] into Polynomial[Interval[A]].

```scala
def translate(p: Polynomial[A])(implicit A: Field[A]) = {
  val ts = p.terms.map {
    case Term(c, e) =>
      Term(Interval.point(c), e)
  }
  val p: Polynomial[Interval[A]] = Polynomial(ts)
  p(this)
}
```

Then we evaluate the new polynomial with this.
intervals as souped-up ranges
ranges

In addition to all this Interval[_] is an acceptable range.

Translation:

• (x to y) becomes Interval(x, y)
• (x until y) becomes Interval.openUpper(x, y)

Caveat: if you're looping over ints, stick with Range (or cfor).
advantages

You could just use NumericRange[_]; why Interval[_]?

1. No need to specify a step up-front.
2. Very fast contains() checks.
3. Math operations (e.g. +)
4. Set operations (e.g. intersection)
5. Overflow checking via NumberTag[_]
6. infinite loop (TM) support

Let's look at some examples...
val ns = Interval(0, 100)
ns.foldOver(0, step = 1)(_ + _) // 5050

val x = ns.iterator(1).sum // 5050

ns.loop(25)(println)
// 0
// 25
// 50
// 75
// 100
getting fancier

We can use union/intersection to encode loop range calculations:

```scala
val x = Interval(x, y)
val y = Interval(w, z)
val z = (loop1 & loop2)

z.loop(step) { ... }
```
how often have you wanted this?

// divide the interval [0, N] into M distinct parts.
// return boundaries as a list.
def divide(n: Int, m: Int): List[Double] = {
  val h = Interval(r"0", Rational(n))
  val step = Rational(n, m)
  h.iterator(step).map(_.toDouble).toList
}

divide(7, 5)  // List(0.0, 1.4, 2.8, 4.2, 5.6, 7.0)
divide(13, 4) // List(0.0, 3.25, 6.5, 9.75, 13.0)
divide(3, 7)  // List(0.0, 3.25, 6.5, 9.75, 13.0)
divide(10, 3) // List(0.0, 3.3333333333333335, 6.666666666666667, 10.0)
how often have you wanted this?

Works on very hairy fractions too!

divide(3, 7)
  // List(0.0,
  //    0.42857142857142855,
  //    0.8571428571428571,
  //    1.2857142857142858,
  //    1.7142857142857142,
  //    2.142857142857143,
  //    2.5714285714285716, 3.0)
infinite loops

Interval.above(r"0").loop(r"1") { n =>
    println(s"$n bottles of beer on the wall")
}
cool demo
caveats

There are a few things to keep in mind when working with `Interval[_]`:

1. It is unspecialized (re loops).

2. `Order[_]` needs to obey additional laws when used with rings.

3. Possible rounding issues with `Double`.

4. Interpretation of `contains` and `isEmpty`.

5. The "dependency problem"
why unspecialized?

In most cases it is more efficient for `Interval[_]` to box.

This means that using `Interval[Int]` purely for looping will be slower than code that avoids boxing altogether.

Prefer `while`, Spire's `cfors`, Scalary/Loops, Range or similar.

[1] Fields in specialized classes are duplicated, and often fail to specialize access correctly.
additional order laws?

Our intervals require \texttt{Order[\_]}, a total ordering.

However, when using \texttt{Order[\_]} with \texttt{Field[\_]} it must:

1. preserve the \textit{algebraic structure}, i.e.
2. \texttt{if } \texttt{a }\leq\texttt{ b then } \texttt{a + c }\leq\texttt{ b + c } and
3. \texttt{if } \texttt{0 }\leq\texttt{ a } and \texttt{0 }\leq\texttt{ b then } \texttt{0 }\leq\texttt{ a * b}

e.g. \texttt{Interval[\_]} cannot work correctly with \texttt{Complex[\_]}. 
but aren't complex intervals possible?

Yes, but Interval[!] doesn't support them.

If you need these, it's a great opportunity to contribute! 😊
rounding issues

1. The JVM doesn't support floating point rounding modes
2. Even if it did, using them would be a global side-effect.
3. Correct approach:
   * use java.lang.Math.{nextUp, nextAfter}
   * But quite ugly to bake this in for generic types

We probably need to extend the NumberTag to help support this.

 Mostly I just use Interval[Rational] instead. 😐
contains and isEmpty

One potential puzzler is this:

```scala
val e = Interval.empty[Int]   // (Ø)
val h = Interval.open(3, 4)   // (3, 4)

e.isEmpty                       // true
(e.contains(3), e.contains(4))  // (false, false)

h.isEmpty                       // false
(h.contains(3), h.contains(4))  // (false, false)
```

What gives?
contains and isEmpty

Remember: A is an approximation of the reals.

Even though Long has no values between 3 and 4, they exist.

Consider:

```scala
val ee = e * 2  // (Ø)
val hh = h * 2  // (6, 8)
hh.contains(7) // true
```

So h was never empty at all!
dependency problem

Here's something strange:

```scala
val h = Interval(-3, 3)  // [-3, 3]
h ** 2  // [0, 9]
h * h  // [-9, 9]
```

What's going on?
dependency problem

If you take one value from \([-3, 3]\) and square it, the smallest value you can get is 0 (e.g. \(0 ** 2\)).

However, if you take two values from \([-3, 3]\) and multiply them, you could get -9 (e.g. \(-3 * 3\)).

This is called the dependency problem.

These results are a correct interval, but could be more precise (i.e. tighter bounds).
conclusion

Hopefully I've helped explain what intervals are.

I believe there are many problems where intervals could be useful.

(Even if it wouldn't be worth it to stop and implement them then.)

Let us know what you think! 📞
there's even more

Learn more about Spire's various types in the Spire User's Guide:

https://github.com/non/spire/blob/master/GUIDE.md

If you want to contribute to Spire, we also have:

https://github.com/non/spire/blob/master/CONTRIBUTING.md
AUTHORS.md

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*Thanks!*
the end